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Quantum Relativity Theory and Quantum Space-Time

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A quantum relativity theory formulated in terms of Davis' quantum relativity principle is outlined. The first task in this theory as in classical relativity theory is to model space-time, the arena of natural processes. It is shown that the quantum space-time models of Banai introduced in another paper is formulated in terms of Davis' quantum relativity. The recently proposed classical relativistic quantum theory of Prugovečki and his corresponding classical relativistic quantum model of space-time open the way to introduce, in a consistent way, the quantum space-time model (the quantum substitute of Minkowski space) of Banai proposed in the paper mentioned. The goal of quantum mechanics of quantum relativistic particles living in this model of space-time is to predict the rest mass system properties of classically relativistic (massive) quantum particles ("elementary particles"). The main new aspect of this quantum mechanics is that provides a true mass eigenvalue problem, and that the excited mass states of quantum relativistic particles can be interpreted as elementary particles. The question of field theory over quantum relativistic model of space-time is also discussed. Finally it is suggested that "quarks" should be considered as quantum relativistic particles.

1. INTRODUCTION

Six years ago M. Davis (1977) established a relativity principle in quantum (q) theory. He interpreted complete Boolean algebras of projections in a Hilbert space as Boolean reference frames relative to which q measurements are made. Then he showed that the formalism of q theory is interpretable in terms of this relativity principle; furthermore he suggested that the quantization of a classical (c) theory means nothing other than the application of appropriate Boolean valuation to the sentences of the theory,

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according to the work of G. Takeuti (1978). As a suggestion for further work he posed the following: "If quantum theory embodies a relativity principle, it surely interacts with the other basic relativity principles. Thus, there should be a relativity theory combining special relativity and quantum theory in which the underlying group combines (perhaps as a direct product) the Lorentz and the unitary groups."

According to this discovery and hypothesis one may expect that a relativity theory can be constructed, which uses, as basic principle, the relativity principle of q theory (we call this principle q relativity principle, while the relativity principle of Einstein c relativity principle), the space-time in this q relativity theory should be built up by applying the q relativity principle. Then this q relativistic space-time should combine with the special relativistic space-time (e.g., in a way as indicated in the hypothesis) and the results of the corresponding hybrid particle mechanics should manifest themselves in the elementary particle phenomenology in a clear way.

We discuss in this paper the possibility of this q relativity theory with the corresponding q models of space-time based on q relativity principle, a concrete q space-time model, one particle mechanics in this q space-time, and field theory over this q space-time. In connection with the Davis guess we present a "superrelativistic" (Davis, 1977, p. 896) particle model in which the underlying space-time combines Minkowski space-time $\mathbb{M}^4$ and this q space-time, the symmetry group is a direct product of the Poincaré and unitary groups. The corresponding hybrid particle mechanics finds application in hadron physics with a good experimental support.

We note that a more detailed exposition of the theses of this paper can be found in Banai (1983a).

2. QUANTUM SPACE-TIME MODELS BASED ON QUANTUM RELATIVITY PRINCIPLE

Let H be a Hilbert space with inner product $\langle | \rangle$. Then $\mathscr{P} = \mathscr{P}(H)$ denotes the lattice of closed linear subspaces of H and $\mathscr{B}$ denotes a complete Boolean sublattice of $\mathscr{P}$. The q relativity principle of Davis involves the following concepts: Boolean frames $\mathscr{B}$ in $\mathscr{P}$ as q equivalent of inertial frames, "q inertial frames"; unitary transformations in H as coordinate transformations between q inertial frames; the unitary group of H as the symmetry group of q coordinate transformations; state vectors $\psi \in H$ as q equivalent of events, "q events"; the Hilbert space H as the q substitute of c event space $\mathbb{M}^4$, "q event space."

Now, in Banai (1983b) we outlined an axiomatic framework to approach general space-time models, aiming at to resolve, at least partly, the deficiency in q logical approach to q theory that these approaches completely lack models of space-time geometry as was noted by Finkelstein (1981a). We called, in this framework, those models of space-time to which irreducible q propositional systems of Piron (1976) belong as (relativistic) causal logic, *q space-times*. According to this approach a space-time model is determined by the corresponding causal logic l; events in the space-time model are the atoms of l, symmetries are generated by automorphisms of l, the event space is the space which realizes l via its "power structure," two events are causally connected (disconnected) if they are noncompatible (compatible). When l is an irreducible q propositional system then l is realizable, disregarding some exceptional cases, by a $\mathscr{P}(H)$ where H is a (generalized) Hilbert space (Piron, 1976). Events are represented in H by rays and, if H is a complex Hilbert space of at least dimension 3, symmetries are generated by unitary (or anti-unitary) operators in H. Two events are causally connected (disconnected) if $\langle\psi_1|\psi_2\rangle \neq 0$ $(=0)$ for the vectors in the corresponding rays. Now, if we interpret complete Boolean algebras $\mathscr{B}$ in $\mathscr{P}(H)$ as Boolean reference frames ("q reference frames") then we see that the q space-time models in Banai (1983b) can be formulated in terms of q relativity principle of Davis. Therefore we can call these space-time models *q relativistic space-time models.*[2]

One can easily verify that the main concepts of c relativity theory, such as events, locality, particle, covariance, and invariance, have clearcut representatives in q relativity theory (see Banai, 1983a). We consider briefly the particle concept only.

As q objects, the particles cannot be viewed as material points following a world line, but rather, in keeping with Heisenberg's (1960) and Wigner's (1939) observations, as constructs empirically associated with operational procedure for measuring position and momentum, and theoretically associated with irreducible representations of the symmetry group of the applicable relativity. In q relativistic physics, this is Davis' relativity with the unitary group G as symmetry group. If the space defining G is the (complex) Hilbert space H then H carries trivially an irreducible representation of G. If the system of self-adjoint operators (P, X) in H is an irreducible system then the self-adjoint generators of G are the functions of

[2]Note that the set of atoms of a maximal Boolean algebra in $\mathscr{P}(H)$ determines a maximal set of causally disconnected events, because it is generated by a complete orthonormal basis of H. Thus such a set of atoms determines a "spacelike hypersurface" in the corresponding q relativistic space-time (Banai, 1983b).

(P, X), and the basic observables of the q particle are represented by the operators (P, X). Any other observable of the q particle is a function of (P, X). The q event space H describes the set of possible coordinates of the q particle with respect to a particular q reference frame and G describes the possible geometrical symmetries of the q particle (cf. Finkelstein, 1981b). (On the analogy of $\mathbb{M}^4$, we may call H the *configuration space* of the *q particle*.) The *arena of the existence* for the c particle is the *manifold* $\mathbb{M}^4$, while the *arena of existence* for the q particle is the projective geometry represented by H.

We note that the universe $V^{(\mathscr{P})}$ of the $\mathscr{P}$-valued model (or q set theory) of Takeuti (1981) involves all Boolean reference frames in $\mathscr{P}$. Thus the appropriate language of q relativity theory we advocate in this paper may be provided by Takeuti's q set theory (cf. Banai, 1982b, 1983a,c).

3. THE QUANTUM RELATIVISTIC SUBSTITUTE OF MINKOWSKI SPACE

We present now a concrete q space-time model. This is the q relativistic substitute of $\mathbb{M}^4$ proposed in Banai (1983b). It is important to note that the results and the *cq* Minkowski space-time model of Prugovečki (1983) open the way to a consistent introduction and interpretation of this q space-time model. Because, as q mechanics presupposes the existence of its c limit, c mechanics (see Landau and Lifshitz, 1958), in a similar way q relativistic q space-time and mechanics in it presupposes the existence of their consistent *cq* counterparts, *cq* space-time, and *cq* mechanics, respectively.

To start the discussion, let us consider the following three concepts of physics: (a) pointlike particle (object), (b) Lorentz inertial frames and hence Lorentz transformations, (c) Heisenberg uncertainty principle. There are growing evidences that these three concepts are operationally not compatible (cf. Prugovečki, 1978, 1981; Kaiser, 1981). One way out of this conceptual difficulty is the rejection of concept (a) and hence the retention of concepts (b) and (c). This attitude was chosen, e.g., by E. Prugovečki (1981, 1983, for a review). He replaced concept (a) by a stochastically extended particle concept. The second way is the drop of concept (c), this attitude was chosen, e.g., by Snyder (1947) (see Banai, 1983a). The third way out of this impasse is the rejection of concept (b), this attitude was chosen by Banai (1983b). According to q relativity theory we replace the concept of Lorentz frames by that of q reference frames.

E. Prugovečki (1983) unifying in a consistent way c relativity and q theory in terms of stochastic spaces found that a c relativistic q particle (*cq* particle or elementary particle) must be characterized not only by its rest mass m, charge e, spin s etc., but by a proper wave function φ, too. This $\varphi(\mathbf{x})$ is a rotationally invariant real element of the Hilbert space $L^2(\mathbb{R}^3)$ and describes the spatial extension of the *cq* particle in its rest Lorentz frame L, placed at the origin of L at the instant $q^0 = 0$. In this framework, *cq* "space-time is envisaged as consisting of "points" which can be occupied by (such) extended elementary particles, and therefore can be envisaged as represented by the proper wave functions of such particles" (Brooke et al., 1982, p. 1731). But the proper wave functions φ are elements of the Hilbert space $L^2(\mathbb{R}^3)$ which represents a projective geometry. Hence, the "points" of *cq* space-time can be represented by appropriate points of this projective geometry. Thus we surmise that if we dig further into the microscopic domain of space-time, then we find there this projective geometry as an underlying geometrical structure of space-time, which gives rise, as higher level constructs, in order to Prugovečki's space-time model in the *cq* level, and to Einstein's model in the *c* level. This projective geometry can be considered as the event space of a *q* relativistic space-time model. Because $L^2(\mathbb{R}^3)$ can be considered as the configuration space of a *q* relativistic *q* object (particle) we can probe the microscopic structure of space-time in the *q* relativistic *q* level (simply *q* level) via the observation of such a *q* object. The *cq* particles of Prugovečki should be the excited states of this *q* particle, i.e., the proper wave functions of the *cq* particles should be associated with different "excited states" of this *q* particle.

The *declared goal* of this theory is to *probe* the *microscopic structure* of space-time via the measurements of a *q* particle with space $L^2(\mathbb{R}^3)$ which is pointlike and obeys Heisenberg's uncertainty principle. Therefore, according to the basic tenets of *q* theory (cf. Sachs, 1982), we collect all the measuring apparatuses corresponding to the Boolean reference frames in $\mathscr{P}(L^2(\mathbb{R}^3))$ into one, arbitrary, but fixed, Lorentz frame L. According to the results of Prugovečki (1983) above, we will interpret L as the *c* rest frame of the *cq* particles generated by the rotationally invariant states of the *q* particle. If we implement a measurement on the *q* particle with respect to one of its Boolean reference frames at the instant $q^0 = 0$ in L and we find that the *q* particle is in the rotationally invariant state φ then we say that we observed a *cq* particle with proper wave function φ which was at rest at $q^0 = 0$ in L. The *cq* particle is described in another Lorentz frame with the corresponding element of its state space $L^2(\Sigma_m^+, \varphi)$ (see Prugovečki, 1981, 1983).

Let us consider the observables of the *q* particle. The irreducible system of self-adjoint operators in $L^2(\mathbb{R}^3)$ is $(\mathbf{x}\cdot, -i\nabla_{\mathbf{x}})$. We interpret this system as the basic 3-position $\hat{\mathbf{x}}$ and 3-momentum $\hat{\mathbf{p}}$ of the *q* particle, respectively, Heisenberg's uncertainty principle then is obviously satisfied.

All other observables are functions of these operators. In the procedure of defining other main observables of the q particle (such as the time coordinate, the energy, the mass etc.) we use the rotational invariance as a natural guiding principle because of our interpretation. Let us see the 3-velocity observable $\hat{\mathbf{u}}$. All possible cq particles generated by this q particle are at rest in L at the instants of their observation. Therefore we *postulate* that the *q particle* is *at rest* in L and thus relative to all of the Boolean reference frames in L too. Then

$$\hat{\mathbf{u}} = 0 \tag{1}$$

But the relation $\mathbf{p} = m\mathbf{u}$ satisfied by both the c particle and cq particle then holds no longer true anyway. We must associate a *mass observable* $\hat{m}$ with the q particle, which is the function of $\hat{\mathbf{x}}$ and $\hat{\mathbf{p}}$, i.e., we cannot imagine this q particle as a massive point with mass m but rather as a pointlike q object which can have different mass states.

Before considering the mass observable $\hat{m}$ let us see the coordinate time observable $\hat{x}_0$ and the canonical energy observable $\hat{p}_0$ of the q particle.[3] By definition $\hat{p}_0$ generates the translations on the spectrum of $\hat{x}_0$, and vice versa, which means that

$$[\hat{p}_0, \hat{x}_0] = \mp i \tag{2}$$

Both signs yield the same Dirac's uncertainty principle

$$\Delta p_0 \Delta x_0 \geqslant \tfrac{1}{2}\hbar \tag{3}$$

which is the physical content of (2). For this reason we consider simultaneously the two cases in Banai (1983b). We note that the proper time[3] $\hat{x}_0$ and 3-position $\hat{\mathbf{x}}$ of the q particle can be in a consistent way related to the internal 4-position $(Q_0, \mathbf{Q})$ of the cq particles generated by the q particle, where Q satisfy the c relativistic CCR $[Q_\mu, P_\nu] = -ig_{\mu\nu}$ (see Prugovečki, 1982), as follows. Let the q particle be in the rotationally invariant state ϕ which then defines a cq particle resting at L. Then the uncertainties

$$\Delta x_i = \delta(x_i, \phi)\left(= \langle\phi|(\hat{x}_i - \bar{x}_i)^2\phi\rangle^{1/2}, \quad \bar{x}_i = \langle\phi|\hat{x}_i|\phi\rangle\right), \quad \Delta p_i = \delta(p_i, \phi)$$

are the same as the uncertainties ΔQ_i and ΔP_i established in the internal

[3]Following from our postulate, $\hat{x}_0$ may be called the proper time observable of the q particle. These two notions, i.e., the coordinate time and proper time, would coincide in this q relativistic space-time model.

3-position and 3-momentum of the cq particle corresponding to ϕ by an observer that travels with the cq particle. And the uncertainties $\Delta x_0 = \delta(\hat{x}_0, \phi)$ and $\Delta p_0 = \delta(\hat{p}_0, \phi)$ are equal to the uncertainties Δq_0 and ΔP_0 displayed by the internal time operator Q_0 and energy operator P_0 of the cq particle in the mean value q_0 and P_0 of the (stochastic) time and energy, respectively, measured by an observer traveling with the cq particle (cf. Prugovečki, 1982). Thus Δx_0 can be interpreted as the lifetime of the state ϕ and hence the proper life-time of the cq particle corresponding to ϕ.

By means of $\hat{p}_0$ we can define the time derivative of an observable $\hat{F}$ of the q particle, according to q theory, as follows:

$$d\hat{F}/d\hat{x}_0 := i[\hat{p}_0, \hat{F}] \tag{4}$$

For example, the 3-velocity is, according to its c definition,

$$\hat{\mathbf{u}} := d\hat{\mathbf{x}}/d\hat{x}_0 = i[\hat{p}_0, \hat{\mathbf{x}}] = 0 \tag{5}$$

taking into account our postulate (1). Then $\hat{p}_0$ must be the function of $\hat{\mathbf{x}}$. The simplest rotationally invariant choice for $\hat{p}_0$ is

$$\hat{p}_0 = \hbar^{-1}(x_i x_i \cdot)^{1/2} = \hbar^{-1} \cdot \hat{r} \tag{6}$$

where $\hbar$ is a constant of dimension $(\text{length})^2 = \text{GeV}^{-2}$ using natural units in this paper (cf. Banai and Lukàcs, 1983a). Then we can satisfy (3) with the rotationally invariant combination

$$\hat{x}_0 = \pm i\hbar \frac{1}{r}(x_i \partial/\partial x_i + 1) \tag{7}$$

of the operators $\mathbf{x}\cdot$ and $-i\nabla_{\mathbf{x}}$,[4] and we observe that

$$[\hat{x}_0, \hat{r}] = \pm i\hbar \tag{8}$$

This commutation relation yields the following uncertainty principle

$$\Delta x_0 \Delta r \geqslant \tfrac{1}{2}\hbar \tag{9}$$

[4]The mathematical subtlety as to the self-adjointness of $\hat{x}_0$ in (7) is studied in details in Banai (1983b) and found that $\hat{x}_0$ can be represented with a bona fide self-adjoint operator which is a rotationally invariant combination of $\mathbf{x}\cdot$ and $-i\nabla_{\mathbf{x}}$ and has a form similar to the symmetric operator in (7), which in spherical coordinates has the form $\hat{x}_0 = \pm i\hbar(1/r)(\partial/\partial r)r = \pm i\hbar(\partial/\partial r + 1/r)$, i.e., it is proportional to the radial component of the 3-momentum $\hat{\mathbf{p}} = -i\nabla_{\mathbf{x}}$.

Exactly this principle, respectively, relation (8) is our starting point in constructing this q space-time in Banai (1983b). The principle (9) means that the q particle under consideration is not localizable in relation to the fixed L better than permitted by (9).[5] We see from (6) that the energy of the q particle rises linearly with its radial distance from the origin of L involved in the empirical definition of the q particle. This origin fixing is also reflected in relation (8) which is invariant under time translations but singles out the origin of L. This is an immediate consequence of the operational definition of the q particle (cf. Banai, 1983a). This feature of defining the q particle can be grasped the most striking way when we consider the time derivative of $\hat{\mathbf{p}}$ and its radial component $\hat{p}_r$, which by c analogy gives the force and its radial component, respectively, acting on the *free* q particle. They are

$$\hat{\mathbf{f}} := d\hat{\mathbf{p}}/d\hat{x}_0 = i[\hat{p}_0, \hat{\mathbf{p}}] = -\hbar^{-1}\hat{\mathbf{x}}/r \tag{10}$$

$$\hat{f}_r := d\hat{p}_r/d\hat{x}_0 = i[\hat{p}_0, \hat{p}_r] = -\hbar^{-1} \tag{11}$$

Consequently, a *constant force acts on the free q particle* in this q space-time, forcing the q particle to the *origin of its defining frame L*. Equation (10) means that the 3-momentum $\hat{\mathbf{p}}$ of the q particle is not conserved with respect to its time $\hat{x}_0$, while the 3-momentum of the cq particles generated by the states of the q particle is, of course, conserved with respect to their time q^0 (cf. Banai, 1983a).

Now we have the time and space position observables $\hat{x}_0$ and $\hat{\mathbf{x}}$ of the q particle. They commute according to

$$[\hat{x}_\mu, \hat{x}_\nu] = \pm i\hbar \hat{A}_{\mu\nu}, \qquad \hat{A}_{\mu\nu} = \frac{1}{r}(\hat{\mathbf{x}}, \mathbf{0}) \tag{12}$$

Let us consider the square of the space-time position observable $\hat{x} = (\hat{x}_0, \hat{\mathbf{x}})$. In c relativity this is $x^2 = g_{\mu\nu}x^\mu x^\nu$. The indefinite metric $g_{\mu\nu}$ follows from the causal relation of the points (representing c events) in $\mathbb{M}^4$. In q relativity the causal relation of the rays (representing q events) in $L^2(\mathbb{R}^3)$ is expressed by the relation $\langle\phi_1|\phi_2\rangle$. ϕ_1 and ϕ_2 are causally connected if $\langle\phi_1|\phi_2\rangle > 0$ and disconnected when $\langle\phi_1|\phi_2\rangle = 0$. Following from the positive definiteness of the metric of $L^2(\mathbb{R}^3)$, we have $\langle\phi|\phi\rangle = \|\phi\|^2 > 0$ for every vector $\phi \neq 0$ in $L^2(\mathbb{R}^3)$, i.e., there are no "spacelike" (or nonzero

[5] Relation (9) can still be read as follows. The smaller the Δr, the greater the Δx_0 and vice versa, i.e., if the ϕ for which Δr means the spread of $\hat{r}$ is highly concentrated near to a point then the lifetime of ϕ is nearly infinite. In other words, the less smeared the cq particle corresponding to ϕ, the longer its lifetime.

"timelike") vectors in the event space $L^2(\mathbb{R}^3)$. These notions have no meaningful counterparts in q relativity (with Hilbert spaces of positive definite metric). The indefinite metric of c relativity should appear first on the cq level of microscopic world which lies over the q level of nature (according to the philosophical suggestion of this paper, cf. Table I and Section 7). [The photon mediating the electromagnetic interaction is also a cq particle. The properties of this cq particle, respectively, interaction play a central role in the operational definition of the Lorentz symmetries and hence of the causal relation of c events. Thus q relativity theory should produce in a natural manner the photon as a cq particle to obtain the symmetries of the cq level and c level of nature (cf. Section 7).] Therefore if

TABLE I

Classical relativistic classical level	Deterministic geometry (c diff. manifold) c reference frame (deterministic Lorentz frame) c relativity (deterministic Poincaré group) c particles (mass points along deterministic world lines)
	c mechanics (deterministic equations for motion with parameter mass and continuous energies) c field theory (deterministic fields over a single c space-time)
Classical relativistic quantum level	Stochastic geometry (cq diff. manifold) cq reference frame (stochastic Lorentz frame) cq relativity (stochastic Poincaré group) cq particles ("elementary particles," photons, W bosons)
	cq mechanics (energy eigenvalue problem with parameter mean time and parameter deterministic mass) cq field theory (quantized fields over a single cq space-time)
Quantum relativistic quantum level	Projective geometry (q manifold) q reference frame (Boolean frame) q relativity (unitary group) q particles ("quarks," "gluons") q mechanics (mass eigenvalue problem with operator time)
	q field theory [(1) q fields (depending on the coordinate time-space operators) inside a single q space-time) (2) canonically quantized fields (depending on the vectors of events) over a single q space-time]

we define the square of $\hat{x}$ as $\hat{x}^2 = g_{\mu\nu}\hat{x}^\mu\hat{x}^\nu = \hat{x}_0^2 - \hat{x}_i\hat{x}_i$ then in the spectrum of $\hat{x}^2$ could appear negative values which would not have clearcut meaning. From these reasons the most natural (and symmetric) choice for defining the square of $\hat{x}$ is

$$\hat{x}^2 \coloneqq \delta_{\mu\nu}\hat{x}_\mu\hat{x}_\nu = \hat{x}_0^2 + \hat{x}_1^2 + \hat{x}_2^2 + \hat{x}_3^2 \tag{13}$$

i.e., with *positive definite metric* $\delta_{\mu\nu} \doteq (1,1,1,1)$ which then guarantees the nonnegativity of the spectrum of $\hat{x}^2$ and thus the possibility of interpreting $\hat{x}^2$ as the $(\text{length})^2$ observable for the q space-time position of the q particle. Then, by (7), (13) gives a purely discrete positive spectrum $x_n^2 = 4\hbar n + 3$, $n = 0, 1, \ldots$, for $\hat{x}^2$ (Banai, 1982a), which implies that the q particle can take only discrete space-time position values in q space-time, i.e., it moves ("jumps") on a lattice, in the $x_0 - r$ plane, with spacing $(2\hbar)^{1/2}$. Note that the negative sign of (2) hence the positive sign of (8) conforms to the positive definiteness of (13) (Banai, 1983b).

The energy and momentum observables $\hat{p}_0$ and $\hat{\mathbf{p}}$ of the q particle commute according to

$$[\hat{p}_\mu, \hat{p}_\nu] = i\hbar^{-1}\hat{A}_{\mu\nu} \tag{14}$$

Let us define the square of the "4-momentum" observable $\hat{p} = (\hat{p}_0, \hat{\mathbf{p}})$, from the same reasons as in the case of $\hat{x}$, as follows

$$\hat{p}^2 \coloneqq \hat{p}_0^2 + \hat{p}_1^2 + \hat{p}_2^2 + \hat{p}_3^2 = \delta_{\mu\nu}\hat{p}_\mu\hat{p}_\nu \tag{15}$$

In c relativity $p^2 = m^2$, therefore, using this analogy, we associate the mass square observable $\hat{m}^2$ of the (spin-0) q particle with $\hat{p}^2$ (cf. Banai and Lukács, 1983a), i.e.,

$$\hat{m}^2 \coloneqq \hat{p}^2 = \hat{p}_0^2 + \hat{p}_i\hat{p}_i \tag{16}$$

This, by (6), implies purely positive discrete spectrum for the mass square of the q particle, namely,

$$m_n^2 = 2\hbar^{-1}(2d + l + \tfrac{3}{2}) = 2\hbar^{-1}(n + \tfrac{3}{2}), \qquad n = 2d + l = 0, 1, 2, \ldots \tag{17}$$

In this way we have defined the main observables of the q particle, namely, the proper time[3] $\hat{x}_0$, the energy $\hat{p}_0$ and mass square $\hat{m}^2$ of the q particle.

4. QUANTUM MECHANICS OF QUANTUM RELATIVISTIC PARTICLES

According to the preceding interpretation the *goal* of q mechanics of this q particle is to *predict* the *rest frame properties of cq particles* including the rest mass, too. Equation (10) may be regarded by c analogy as the *equation of motion* for the *free* scalar q particle. The pure states of the q particle coincide with the events of q space-time. The time evolution of the q particle is generated by the unitary map $\phi \mapsto \phi' = U(x_0)\phi = \{\exp[-ix_0\hat{p}_0]\}\phi$ of $L^2(\mathbb{R}^3)$ onto itself, where $x_0 \in \mathbb{R}$. The cq Klein–Gordon equation now is replaced by the $(\text{mass})^2$ eigenvalue equation

$$\hat{p}_0^2\phi + \hat{p}_i\hat{p}_i\phi = m^2\phi, \qquad \phi \in L^2(\mathbb{R}^3) \tag{18}$$

which gives the $(\text{mass})^2$ spectrum (17) for the *free* q particle. The interaction of the q particle with an external field given classically by a 4-vector potential $A_\mu = A_\mu(x_0, \mathbf{x})$ can be described by applying the usual algorithm. In $A_\mu(x_0, \mathbf{x})$ x_0 and $\mathbf{x}$ are replaced by $\hat{x}_0$ and $\hat{\mathbf{x}}$, respectively, satisfying (12) and summations with $g_{\mu\nu}$ in A_μ are converted into summations with $\delta_{\mu\nu}$. The canonical "4-momentum" observable of the system from its c counterpart is $\hat{P}_\mu = \hat{p}_\mu + eA_\mu(\hat{x}_0, \hat{\mathbf{x}})$ satisfying (14). Then $\hat{p}_\mu = \hat{P}_\mu - e\hat{A}_\mu$ and the $(\text{mass})^2$ eigenvalue equation (18) reads as follows:

$$\left(\hat{P}_\mu - e\hat{A}_\mu\right)\left(\hat{P}_\mu - e\hat{A}_\mu\right)\phi = m^2\phi, \qquad \phi \in L^2(\mathbb{R}^3) \tag{19}$$

(summation for μ is understood with $\delta_{\mu\nu}$, of course). For rotationally invariant A_μ, according to our interpretation, an eigenstate ϕ_n with eigenvalue m_n^2 may be regarded as the proper wave function of a cq particle with mass m_n, i.e., we associate with ϕ_n the c relativistic wave function $\tilde{\eta}_n(k) = m_n^{1/2}\tilde{\phi}(k)$, $k = (+(\mathbf{k}^2 + m_n^2)^{1/2}, \mathbf{k})$, which then generates, by Prugovečki's (1983) results, the Poincaré-invariant subspace $L^2(\Sigma_{m_n}^+, \eta_n)$ in the Hilbert space $L^2(\Sigma_{m_n}^+)$. Thus, e.g., the cq particles corresponding to the $(\text{mass})^2$ spectrum (17) of the *free* scalar q particle lie along linear Regge trajectories. Consequently, they may be identified as meson states in the cq particle phenomenology.

Up to now we have considered only scalar q particles. For a spinor q particle, using Dirac's idea, we may assume a mass operator $\hat{m}$ linear in momenta. The leading terms are the same as in (16) if

$$\hat{m} = \tilde{\gamma}_\mu\hat{p}_\mu = i\gamma_0\hat{p}_0 + \boldsymbol{\gamma}\hat{\mathbf{p}}, \qquad \tilde{\gamma} = (i\gamma_0, \boldsymbol{\gamma}) \tag{20}$$

where γ_μ's are Dirac's matrices. Then for $\hat{m}^2$ we obtain by (14)

$$\begin{aligned}\hat{m}^2 &:= -\tilde{\gamma}_\mu\hat{p}_\mu\tilde{\gamma}_\nu\hat{p}_\nu = \hat{p}_\mu\hat{p}_\mu - \tfrac{1}{2}\tilde{\sigma}_{\mu\nu}[\hat{p}_\mu, \hat{p}_\nu] = \hat{p}_0^2 + \hat{\mathbf{p}}^2 - (i/2)\hbar^{-1}\tilde{\sigma}_{\mu\nu}\hat{A}_{\mu\nu} \\ &= \hat{p}_0^2 + \hat{\mathbf{p}}^2 - \hbar^{-1}\boldsymbol{\alpha}\hat{\mathbf{x}}/r \end{aligned} \tag{21}$$

where $\tilde{\sigma}_{\mu\nu} = (1/2)(\tilde{\gamma}_\mu\tilde{\gamma}_\nu - \tilde{\gamma}_\nu\tilde{\gamma}_\mu) = (1/2)(i\boldsymbol{\alpha}, i\boldsymbol{\Sigma})$. The new term in $\hat{m}^2$ depends only upon the spin of the spinor q particle (see Banai and Lukács, 1983a). In the case of the interaction, by the substitution $\hat{p}_\mu \to \hat{p}_\mu - e\hat{A}_\mu$ we get the

$\hat{m}^2$ observable

$$\hat{m}^2 = (\hat{P}_\mu - e\hat{A}_\mu)(\hat{P}_\mu - e\hat{A}_\mu) - \tfrac{1}{2}\tilde{\sigma}_{\mu\nu}[\hat{P}_\mu - e\hat{A}_\mu, \hat{P}_\nu - e\hat{A}_\nu]$$

$$= (\hat{P} - e\hat{A})^2 - (i/2)\hbar^{-1}\tilde{\sigma}_{\mu\nu}\hat{A}_{\mu\nu} + (i/2)e\tilde{\sigma}_{\mu\nu}\hat{F}_{\mu\nu} - (e^2/2)\tilde{\sigma}_{\mu\nu}[\hat{A}_\mu, \hat{A}_\nu]$$

$$= (\hat{P} - e\hat{A})^2 + \hbar^{-1}\alpha\hat{\mathbf{x}}/r - e(\Sigma\hat{\mathscr{H}} + \alpha\mathscr{E}) - (e^2/2)\tilde{\sigma}_{\mu\nu}[\hat{A}_\mu, \hat{A}_\nu] \qquad (22)$$

where $\hat{F}_{\mu\nu} = \partial_\mu\hat{A}_\nu - \partial_\nu\hat{A}_\mu = i[\hat{P}_\mu, \hat{A}_\nu] - i[\hat{P}_\nu, \hat{A}_\mu] = (-\hat{\mathscr{E}}, \hat{\mathscr{H}})$. The new terms in $\hat{m}^2$ predict the interaction of the spin of the q particle with the external field. It is worthwhile to note that we started with a non-self-adjoint linear expression (20) for $\hat{m}$ and we obtained a purely self-adjoint $\hat{m}^2$ observable in (21) and (22) (more precisely, (22) is self-adjoint if $\tilde{\sigma}_{\mu\nu}[\hat{A}_\mu, \hat{A}_\nu]$ is self-adjoint). Now the case is reversed than in the case of Dirac's (1928) equation where one starts with a self-adjoint linear expression and then gets, in the case of the interaction, a non-self-adjoint second-order expression.

We can conclude that the major *new aspect* of q mechanics of q particles is that it provides true mass eigenvalue equations for the possible masses of elementary particles. In cq mechanics the central problem is the energy eigenvalue problem; in q mechanics that would be the mass eigenvalue problem. The lifetimes of different mass states and the decay modes of higher mass states into lower ones can be calculated using purely the tools of this q mechanics. Hence the model would be capable of dealing with the corresponding unstable cq particles calculating their lifetimes and decay modes.

Let us consider yet briefly the cq limit "$\hbar \to 0$" of q mechanics in q space-time. Exactly as in cq mechanics the limit "$\hbar \to 0$" is defined (von Neumann 1955), the limit "$\hbar \to 0$" is defined by those elements of $L^2(\mathbb{R}^3)$ for which the product $\Delta x_0 \Delta r$ is minimal, i.e., $\Delta x_0 \Delta r = \frac{1}{2}\hbar$. These elements of $L^2(\mathbb{R}^3)$, up to phase factors, have wave-packet-like form $\phi^{(r)}(\mathbf{x})$ with $r = (\hbar/\gamma)^{1/2}$, $\gamma > 0$. When "$\hbar \to 0$" these q events concentrate on the points of c event space $\mathbb{M}^4$, the corresponding extended cq particles shrink to pointlike particles. Therefore, when we take the formal limit "$\hbar \to 0$," q mechanics over q space-time, via the consistent cq mechanics of Prugovečki (1983), turns into the (not truly consistent) conventional cq mechanics and q space-time, via the cq space-time of Prugovečki (1983), reduces to c space-time (cf. Banai, 1983a, b).

5. A SUPERRELATIVISTIC PARTICLE MODEL

We now present a "superrelativistic" particle model for hadrons on the basis of the work Banai (1982a). In the above considerations the models of space-time follow one another in a hierarchical order. In this simple model we avoid the intermediate *cq* model of space-time, and the *q* model and the *c* model of space-time will simply be juxtaposed. Therefore this model is admittedly not consistent in all respects; nevertheless, it helps to analyze some properties of hadrons and to compare the current quark hypothesis with that of *q* particles in *q* relativity theory. The model is motivated by the *c* relativistic quark models (Feynman et al., 1971; Kim et al., 1979), and reflects the expectation that space-time is *q* in regions of size $\sim \hbar$, while it is *c* in regions of size $\gg \hbar$.

The hadron (meson, for the sake of simplicity) consists of two *pointlike particles*, "quarks," q and $\bar{q}$ with positions x_μ^q and $x_\mu^{\bar{q}}$, respectively, at *c* level. Let X_μ and x_μ denote the center-of-mass (c.m.) coordinates describing the hadron as a whole and the relative coordinates describing the internal motion of quarks, respectively. In the model, we approximate in such a way that the c.m. coordinates remain Lorentz variables and they span the Minkowski space in which the hadron as a whole moves, while the relative coordinates are subjected to the quantization condition (8). Thus the internal space of the hadron becomes *q* space-time in which the quarks move. Then the state function of the hadron has the form $\phi = \phi(X, \mathbf{x})$, i.e., it depends upon seven variables $(X, \mathbf{x})$ in such a way that one can pursue all methods of conventional *cq* theory in the X_μ variables, but one should apply the methods of *q* mechanics discussed above in the $\mathbf{x}$ variables to describe the internal quark dynamics. One can regard the state function $\phi(X, \mathbf{x})$ as a function describing an extended particle in $\mathbb{M}^4$, where the spatial extension of the particle in its rest frame is described by the $\mathbf{x}$ variables. The total symmetry group of the hadron is the direct product of the Poincaré group of the X variables and the unitary group of internal *q* space-time. In this way, this model combines *c* relativity with *q* relativity in the way suggested by Davis (1977).

Now we can regard (6) as that the energy rises linearly with the radial quark separation, which then gives rise to the "confining force" in (10), respectively, in (11). It follows at once that the quarks are permanently confined inside a sequence in time of space-time bubbles of size $\sim \hbar$ around the c.m. world line of the hadron. Considering the internal quark dynamics, we can see that the hadronic mass spectrum is generated by the internal quark mass level excitation. We can write for the $(\text{mass})^2$ of the meson, assuming the additivity of mass,

$$M^2 = (m_0 + \hat{m})^2 = m_0^2 + 2m_0\hat{m} + \hat{m}^2 = P^2 + 2(P^2)^{1/2}(\hat{p}^2)^{1/2} + \hat{p}^2 \tag{23}$$

If we suppose that the quarks are *free particles* in internal q space-time, then from (17), we obtain the (mass)^2 spectrum

$$M_n^2 = \left\{ m_0 + \left[2\hbar^{-1}\left(n+\tfrac{3}{2}\right)\right]^{1/2}\right\}^2$$

$$= m_0^2 + 2m_0\left[2\hbar^{-1}\left(n+\tfrac{3}{2}\right)\right]^{1/2} + 2\hbar^{-1}\left(n+\tfrac{3}{2}\right) \qquad (24)$$

If we set $m_0 = 0$ then, in this approximation, the model predicts an equal spacing rule with a spacing $2\hbar^{-1}$, and implies linear Regge trajectories with the same slope $\frac{1}{2}\hbar$ for mesons and baryons. These predictions agree well with the experiment for low-laying hadrons (see Feynman et al., 1971; Kim and Noz, 1972). Now the measured value of the Regge slope gives the following fit for $\hbar$: $2\hbar^{-1} \simeq 1\ \mathrm{GeV}^2$ then $\hbar \simeq 2\ \mathrm{GeV}^{-2} = \frac{2}{5}$ fm/GeV.

In constructing this hybrid model we envisaged at the cq level two cq particles ("quarks") with configuration space $\mathbb{M}^4 \times \mathbb{M}^4$, as the constituents of a meson. Then, in the model, we replaced these two cq particles with one cq particle of mass m_0 and one q particle of mass operator $\hat{m}$, the configuration space of this composed system is $\mathbb{M}^4 \times L^2(\mathbb{R}^3)$.[6] This simple model possesses the basic characteristics of the current quark model of hadrons. Furthermore, it can reproduce the main results of c relativistic quark models, and explain some assumptions in these models [e.g., the equality of the Regge slope for mesons and baryons (see Banai, 1982a)].[7] In the quark model hadrons (cq particles) are envisaged as the excited states of "confined quarks" (also cq particles), in the present theory hadrons are the excited states of q particle (particles).[6] The "confinement" of q particle (particles)[6] inside the hadrons now has a straightforward (purely q theoretical) explanation as we saw above (cf. Section 7).

6. FIELD THEORY OVER QUANTUM SPACE-TIME

Let $\phi_1(x), \phi_2(x), \ldots, \phi_n(x)$ be the n real field variables of a c relativistic local field theory (CRLFT) over $\mathbb{M}^4$. In the Lagrangian framework, this

[6]For baryons, we envisage three quarks with configuration space $\mathbb{M}^4 \times \mathbb{M}^4 \times \mathbb{M}^4$. In the model, in that case, we replace this system with a system consisting of one cq particle with mass m_0 and two q particles with mass operators $\hat{m}_1$ and $\hat{m}_2$, respectively. The corresponding configuration space is $\mathbb{M}^4 \times L^2(\mathbb{R}^3) \otimes L^2(\mathbb{R}^3)$, and the (mass)^2 observable is $M^2 = (m_0 + \hat{m}_1 + \hat{m}_2)^2 = m_0^2 + \hat{m}_1^2 + \hat{m}_2^2 + 2m_0(\hat{m}_1 + \hat{m}_2) + 2\hat{m}_1\hat{m}_2$.

[7]In the construction of the characteristic observables of the q particle we used only the rotational group $O(3)$ from the old space-time symmetry group. In the c relativistic quark model the minimal space-time symmetry group of confined quarks is also the group $O(3)$ as was pointed out by Kim et al. (1982).

theory is given by the Lagrangian density

$$\mathscr{L}(x)=\mathscr{L}(\phi_1,\ldots,\phi_n,\partial_\mu\phi_1,\ldots,\partial_\mu\phi_n)(x),\qquad x\in\mathbb{M}^4 \tag{25}$$

In transferring this theory to a field theory over q space-time, the one possibility is that when we replace the c space-time position vector $(x_0,\mathbf{x})$ in the argument of fields by the q space-time position operator $(\hat{x}_0,\hat{\mathbf{x}})$ satisfying (12). This possibility will be discussed elsewhere. The other possibility is that when we regard the vector x in the argument of fields as denoting an event in $\mathbb{M}^4$ and we replace it by a q event, i.e., in that case we consider the fields as (operator-valued) "functions" $\mathbf{f}\to\phi_\alpha(\mathbf{f})$, $\alpha=1,\ldots,n$, of the rays $\mathbf{f}$ of $L^2(\mathbb{R}^3)$. We discuss now shortly this approach to LFT over q relativistic space-time (QRLFT) according to the works Banai (1982b, 1983a,c) and Banai and Lukâcs (1984) and show that, by applying canonical quantization algorithm, the canonical equations as operator equations are formally equivalent to the c field equations belonging to Lagrangian density (25).

Our guiding principle is the locality: all information obtainable from the system can be gotten by measuring the system at the points of the pertinent space-time model. In CRLFT this principle is formulated in a natural way, i.e., the system is described by local fields and local observables, the global observables are generated by local ones usually by integrating up local observables over spacelike surfaces in $\mathbb{M}^4$. In QRLFT the situation should be the same, i.e., first the local characteristics (local states, observables) of the system should be determined and then the global characteristics generated by local ones by integrating up the local characteristics over spacelike surfaces[2] in q space-time.

In this approach the Hilbert realization of the system of local propositions (Banai, 1981) of QRLFT is determined by means of a Hilbert A module $\mathscr{H}_A$, where A is the von Neumann algebra generated by the projectors of $L^2(\mathbb{R}^3)$ (Banai, 1983c). Thus all informations obtained by local measurements on the system are contained by $\mathscr{H}_A$: the local states can be represented by rays ψ, $\langle\psi|\psi\rangle_A=1$, of $\mathscr{H}_A$ where $\langle\ |\ \rangle_A$ is the A-valued Hermitian inner product of $\mathscr{H}_A$, and the local bounded observables by self-adjoint bounded operators (A-module homomorphisms) in $\mathscr{H}_A$. The expectation value of a local bounded observable F in the local state ψ can be given by the formula

$$\bar{F}=EpF=\langle\psi|F|\psi\rangle_A\in A \tag{26}$$

To determine the concrete structure of $\mathscr{H}_A$ consider the structure of CRLFT of Lagrangian (26). This theory prescribes a c mechanical system at each point x of a spacelike hypersurface σ in $\mathbb{M}^4$. One can think of this system as a "fiber bundle" of c mechanical systems over σ. This "fiber bundle" is of trivial type because (26) prescribes the same c mechanical system to each point of σ. More precisely, according to the approach of Kijowski and Tulczyjew (1979) a phase bundle g: $P \to \mathbb{M}^4$ corresponds to the system. The fiber $P_x = g^{-1}(x)$ is the phase space at $x \in \mathbb{M}^4$. States ϕ of the c fields are sections of the bundle P and P for a Lagrangian CRLFT over $\mathbb{M}^4$ is of trivial type.

Hence the quantized theory should have a similar "trivial fiber bundle" structure, i.e., should describe a q mechanical system at each point p of a spacelike hypersurface[2] Γ in q space-time, the same type of q mechanical system at each point. Thus, if the complex separable Hilbert space $\mathscr{H}$ is the state space of the q mechanical system, the local state space $\mathscr{H}_A$ of QRLFT should have the trivial A-module form $\mathscr{H}_A = \mathscr{H} \otimes A$. In fact, $\mathscr{H} \otimes A$ has the required trivial Hilbert bundle structure over any Γ. For, let $\mathscr{B}$ be any maximal Boolean algebra in $\mathscr{P}(L^2(\mathbb{R}^3))$ with spectrum space Γ and B be the Abelian von Neumann algebra generated by the elements of $\mathscr{B}$. Then the Hilbert B-module $\mathscr{H} \otimes B$ is a subspace of $\mathscr{H} \otimes A$ and isomorphic to the B-module of sections ψ of the trivial Hilbert bundle $\eta: \mathscr{H} \times \Gamma \to \Gamma$ (cf. Dixmier and Douady, 1963). In this sense one may think of $\mathscr{H}_A = \mathscr{H} \otimes A$ as the A-module of sections ψ of the trivial "noncommutative" Hilbert bundle over q space-time of event space $L^2(\mathbb{R}^3)$.[8] In QRLFT the q substitute of the c phase bundle is this "noncommutative" Hilbert bundle. In this way our approach may provide a *natural quantization scheme* for the approach of Kijowski and Tulczyjew (1979) to CRLFT (see Figure 1 and cf. below).

In keeping with our strategy the system globally is described via integrations over the sets of informations obtained by local measurements, i.e., over the local state space $\mathscr{H}_A$. Measures in q space-time are completely known by Gleason's (1957) theorem (Banai, 1983b). Particularly, probability measures are determined by von Neumann density operators $\rho \in A$. In these cases the global state space is given by the Hilbert space $H_\rho := \mathrm{Tr}\rho(\mathscr{H} \otimes A) := \{\phi | \phi \in \mathscr{H} \otimes A, \mathrm{Tr}\rho\langle\phi|\phi\rangle_A < \infty\}$ with scalar product $\langle\phi_1|\phi_2\rangle_\rho := \mathrm{Tr}\rho\langle\phi_1|\phi_2\rangle_A$ (Banai, 1983c). H_ρ describes the q system consisting of an infinite collection of q mechanical systems, *globally* as a q statistical system

[8] Using the Fock representation for $\mathscr{H}$, the vacuum state in the fiber $\mathscr{H}_p = \eta^{-1}(p)$ has the form $\psi_0(p) = \phi_0 \otimes p$ where ϕ_0 is the vacuum state in $\mathscr{H}$. A creation operator $\hat{a}^*_\alpha(p)$, $\alpha = 1, \ldots, n$, in $\mathscr{H}_p$ creates a state $\psi_{1\alpha}(p) = \hat{a}^*_\alpha(p)\psi_0(p)$ in $\mathscr{H}_p$ from this vacuum. This q state at p has q numbers $(Q_1, \ldots, Q_n, q_1, q_2, q_3)$ where (q_1, q_2, q_3) denotes the q numbers of the q event p and interpretable as the internal q numbers of the created q state. The new feature of this approach that the vacuum of local q system is also labeled by these internal q numbers. Having regard to the one particle meaning of q space-time (Sections 2, 3, and 4) we may say that the vacuum is filled up by the states of the q particle and if p is rotationally symmetric then the vacuum $\psi_0(p)$ is labeled by the internal q numbers of a cq particle.

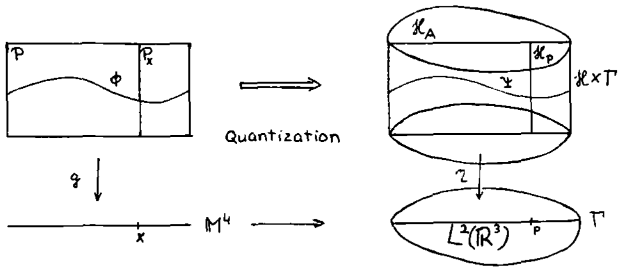

Fig. 1

when the measuring processes of the system is characterized by the density operator ρ (Banai, 1983c). The expectation value of a bounded global observable b (a bounded self-adjoint operator in H_ρ) generated by the local one B in the global pure state ϕ (a ray of H_ρ) is

$$\langle\phi|b|\phi\rangle_\rho = \mathrm{Tr}\,\rho\langle\phi|B|\phi\rangle_A \tag{27}$$

The scalar product $\langle|\rangle_\rho$ is invariant under the action of the unitary symmetry group of q space-time, thus H_ρ is a q relativistically invariant object over q space-time (cf. Banai, 1983c), Banai and Lukács (1984).

In this framework the canonical quantization method can be consistently formulated in terms of the unbounded extension $\mathscr{H}\otimes\hat{A}$ of $\mathscr{H}\otimes A$, where $\hat{A}$ is the *-algebra of linear operators in $L^2(\mathbb{R}^3)$ (Banai, 1983c). According to this algorithm we postulate that the basic variables $\phi_\alpha, \pi_\alpha = \partial\mathscr{L}/\partial(\partial_0\phi_\alpha), \nabla_x\phi_\alpha, \nabla_x\pi_\alpha$, $\alpha=1,\ldots,n$ are local observables represented by self-adjoint operators in $\mathscr{H}\otimes\hat{A}$ such that they satisfy the equal time commutation rules

$$[\hat{\phi}_\alpha, \hat{\phi}_\beta] = [\hat{\pi}_\alpha, \hat{\pi}_\beta] = 0, \qquad [\hat{\pi}_\alpha, \hat{\phi}_\beta] = -i\delta_{\alpha\beta}\mathbb{1}\cdot 1,$$
$$\mathbb{1} = 1_{\mathscr{H}}\otimes 1_H, \qquad 1_H = 1\in A \tag{28}$$

$$[\nabla_x\hat{\pi}_\alpha, \hat{\pi}_\beta] = [\nabla_x\hat{\phi}_\alpha, \hat{\phi}_\beta] = 0, \qquad [\nabla_x\hat{\pi}_\alpha, \hat{\phi}_\beta] = [\hat{\pi}_\alpha, \nabla_x\hat{\phi}_\beta] = -i\delta_{\alpha\beta}\mathbb{1}\hat{\mathbf{p}}$$
$$\hat{\mathbf{p}} = -i\nabla_x \tag{29}$$

The CCR's (28) have an irreducible solution in $\mathscr{H}\otimes\hat{A}$ unique up to A-unitary equivalence, since a natural extension of von Neumann's theorem holds true in this framework (Banai, 1983c). To specify the dynamics the local time evolution is defined by the one parameter unitary group $x_0 \mapsto \exp\{-i\hat{\mathscr{H}}x_0\}$ in $\mathscr{H}\otimes\hat{A}$ where $\mathscr{H} = \mathscr{H}(\phi_\alpha, \pi_\alpha, \nabla_x\phi_\alpha) = \pi_\alpha\partial_0\phi_\alpha - \mathscr{L}$ is the Hamiltonian density. Then the formula $\hat{F}(x_0) = \exp\{-i\hat{\mathscr{H}}x_0\}$ $\hat{F}(0)\exp\{i\hat{\mathscr{H}}x_0\}$ defines the time evolution of local observable $\hat{F}$. This formula in differential form provides the canonical equations

$$\begin{aligned} \partial_0\hat{\phi}_\alpha &= i[\hat{\mathscr{H}}, \hat{\phi}_\alpha] = \partial\hat{\mathscr{H}}/\partial\hat{\pi}_\alpha \\ \partial_0\hat{\pi}_\alpha &= i[\hat{\mathscr{H}}, \hat{\pi}_\alpha] = -\partial\hat{\mathscr{H}}/\partial\hat{\phi}_\alpha - (\partial\hat{\mathscr{H}}/\partial\hat{\nabla}_x\phi_\beta)(\partial\hat{\nabla}_x\phi_\beta/\partial\hat{\phi}_\alpha) \\ &= -\partial\hat{\mathscr{H}}/\partial\hat{\phi}_\alpha - \hat{\mathbf{p}}(\partial\hat{\mathscr{H}}/\partial\hat{\nabla}_x\phi_\alpha) \end{aligned} \tag{30}$$

where (29) has been applied. These canonical equations as operator equations are formally equivalent to the canonical equations of CRLFT [cf. equation (17.7) on p. 118 in Kijowski and Tulczyjew (1979)] obtained in the following way. By applying the c field equations and a Legendre transformation, we get $d\mathscr{L} = [\partial_0\pi + \partial_k(\partial\mathscr{L}/\partial\partial_k\phi_\alpha)]\,d\phi_\alpha + \pi_\alpha d(\partial_0\phi_\alpha) +$ $(\partial\mathscr{L}/\partial\partial_k\phi_\alpha)\,d(\partial_k\phi_\alpha)$, then $d\mathscr{H} = -[\partial_0\pi + \partial_k(\partial\mathscr{L}/\partial\partial_k\phi_\alpha)]\,d\phi_\alpha + \partial_0\phi_\alpha\,d\pi_\alpha$ $-(\partial\mathscr{L}/\partial\partial_k\phi_\alpha)\,d(\partial_k\phi_\alpha)$. Hence we get $\partial_0\phi_\alpha = \partial\mathscr{H}/\partial\pi_\alpha$, $-\partial_0\pi -$ $\partial_k(\partial\mathscr{L}/\partial\partial_k\phi_\alpha) = \partial\mathscr{H}/\partial\phi_\alpha$ and $\partial\mathscr{H}/\partial\partial_k\phi_\alpha = -\partial\mathscr{L}/\partial\partial_k\phi_\alpha$, $k = 1,2,3$, or

$$\partial_0\phi_\alpha = \partial\mathscr{H}/\partial\pi_\alpha, \qquad \partial_0\pi_\alpha = -\partial\mathscr{H}/\partial\phi_\alpha + \partial_k(\partial\mathscr{H}/\partial\partial_k\phi_\alpha) \tag{31}$$

Taking into account the relations $\hat{\nabla}_x\phi_\alpha = \hat{\mathbf{p}}\hat{\phi}_\alpha$ [see Banai, 1983c, equation (19)] the equations (30) are formally equivalent to (31).

Finally we note: (a) The interaction picture can be introduced in this approach in a well-defined manner and the corresponding S matrix is free of divergences. (b) This approach is a natural extension of the conventional Hilbert space formulation of q mechanics. The appropriate language for formulating this theory is provided by Takeuti's (1981) q set theory. Our conjecture is that *q mechanics in the universe $V^{(\mathscr{P})}$, $\mathscr{P} = \mathscr{P}(L^2(\mathbb{R}^3))$, equals QLFT*. The proof of this statement requires, of course, further intensive researches both physically and mathematically (cf. Banai, 1983a,c).

7. CONCLUDING REMARKS

1. We outlined in this paper the foundations of a q relativity theory formulated in terms of the q relativity principle of Davis (1977). As in c relativity theory the first objective of this theory should be the construction of a q relativistic model of space-time. Our basic thesis for approaching this problem is that, according to the philosophical view that *space-time is the arena of the material processes, and of natural phenomena, the different physical theories* with different operational foundations should possess *different models for space-time* (cf. Banai, 1983a). Thus the arena of the

existence for a c relativistic c particle is $\mathbb{M}^4$ and for a c relativistic q particle is the Minkowski q space-time of Prugovečki (1981, 1983). Then this arena for a q relativistic q particle should be the projective space corresponding to its defining Hilbert space H [the q substitute of $\mathbb{M}^4$ in Banai (1983b)]. The main philosophical suggestion of this paper is that these different models of space-time are based on one another in a successive order according to the hierarchical order of the corresponding theories. The deepest level of microscopie world (with respect to our present knowledge) should be approached by q relativity theory. A one more higher level of microworld should be described by c relativistic q theory and the highest level (our macroscopic environment) is approached by c relativity theory. We summarized in Table I this suggestion with the corresponding concepts in the pertinent theory.

2. Certainly the greatest mystery of present-day physics is the quark puzzle. The quark supporters should like to imagine the hadrons as the states of somehow confined quarks. However, this conjecture does not rely on a theory with solid foundations as pointed out by Santilli (1981). As we saw in Sections 3 and 5 q relativity theory might offer a straightforward way for the resolution of this puzzle. One may visualize a q particle at the cq level as a pair of "quarks." Then one could try to model these "quarks" on this same level *assuming* a harmonic oscillator force between the quarks and obtaining in this way the harmonic oscillator model of "confined" quarks. However, the point of our proposal is that, without this assumption, the quark separation space-time coordinates fulfil the "quantization condition" (8) and in this way the quark pair becomes a *single* q particle "*confined*" for ever inside the meson. Moreover, as we saw in Section 3, the operational introduction of such a q particle does not need the assumption of the artificial notion of the quark pair at all. The linear rise of the energy of the q particle in (6) and the ensuing "confining" force in (10) would follow from the very essence of the q particle. In contrast with the "confined" quark hypothesis, i.e., they should be unobservable as individual cq particles, now a *q particle* would be completely *observable*. Because the *observation of a q particle* amounts to the observation of a complete set of its *excited mass states*, i.e., to the observation of a complete set of *cq particles with rest masses* as e.g., indicated by (17), each resting at the origin of the (defining) Lorentz frame L of the q particle at the instant of observation. Therefore we suggest that the "*quarks*" should be in fact identified with *q particles*. Furthermore, the gluons as the mediators of the interactions among "quarks" in the very fashionable "QCD" should be identified with the "(q) *particles*" mediating the interactions among q particles. The pertinent theory of describing the interactions among q particles with the aid of mediators would be the field theory inside q space-time, i.e.,

the "first" quantized version of a c field theory, in which, roughly speaking, the 4-vector $(x_0, \mathbf{x})$ is replaced by the operators $(\hat{x}_0, \hat{\mathbf{x}})$ satisfying the commutation rules (12). We note that one should expect from this theory that produces in a natural way the cq particle photon, in this way indicating how the totally unitary symmetric (infinitely generated) underlying q space-time gives rise to (breaks down) a finitely generated Poincaré symmetric space-time at a higher level description.

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